8-27

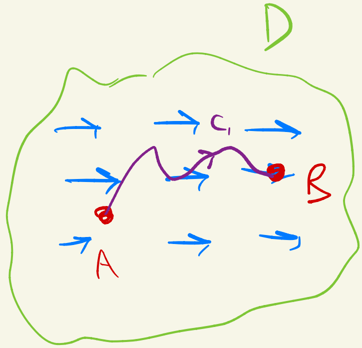
Last time..

- Fund Thm Line integrals:

  ∂S = Boundary of S

$$\int_{C_1} \nabla f \cdot d\vec{r} = f(B) - f(A)$$

boundary ∂C_1

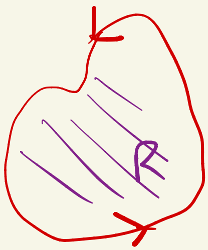


Theme:

$$\iint_R \text{"derivative } F\text{"} \cdot dA = \int_{\partial R} \text{"}F \cdot d\vec{r}\text{"}$$

surface

boundary of surface

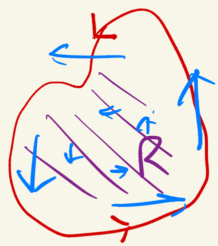


$C = \partial R$

- Green's Theorem
  - Curl-Circulation (Tangential Form)

$C = \partial R$

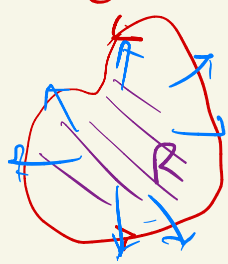
$$\iint_R (\underbrace{\nabla \times \vec{F}}_{\text{curl}\,\vec{F}}) \cdot \hat{k}\, dx\, dy = \oint_{\partial R} \vec{F} \cdot \vec{T}\, ds$$



  - Flux-Divergence (normal form)

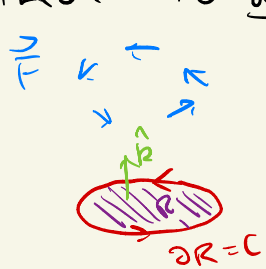
$C = \partial R$

$$\iint_R (\underbrace{\nabla \cdot \vec{F}}_{\text{div}\,\vec{F}})\, dx\, dy = \oint_{\partial R} \vec{F} \cdot \vec{n}\, ds$$



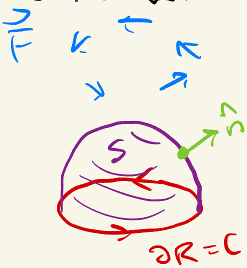
Where we're going:

- Stokes's theorem extends Green's curl-circulation theorem to <u>any</u> surface R with boundary $\partial R = C$:



Green's (tangential)

$$\iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dA = \oint_{\partial R} \vec{F} \cdot d\vec{r}$$



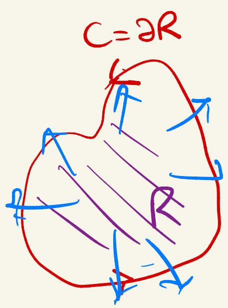
Stokes

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = \oint_{\partial R} \vec{F} \cdot d\vec{r}$$

- Green's Thm (normal)

$$\iint_R (\nabla \cdot \vec{F})\, dx\, dy = \oint_{\partial R} \vec{F} \cdot \vec{n}\, ds$$

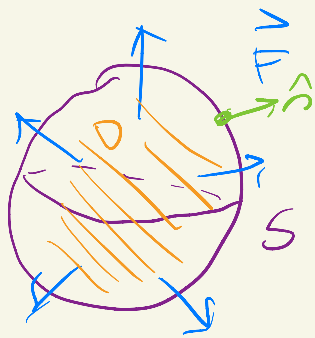
Total divergence inside = outward flux boundary



- Divergence Thm

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, d\sigma$$

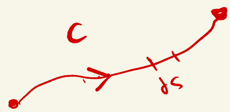
Total divergence inside = outward flux boundary



We need to learn how to compute surface integrals:

Just like line integrals, will need to figure out two things:

- How to parametrize?
  - Or are there alternatives? (sometimes)



$$\vec{r}(t) = \underbrace{f(t)}_{x}\,\hat{i} + \underbrace{g(t)}_{y}\,\hat{j} + \underbrace{h(t)}_{z}\,\hat{k}$$

$a \leq t \leq b$

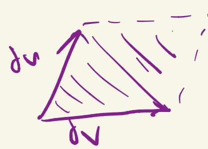
- What's the infinitesimal area?
  - 1D: $ds = |\vec{r}\,'(t)|\,dt$
  - Expect: $d\sigma = |\underbrace{\quad\quad}_{\text{derivatives of } \vec{r}(u,v) \text{ (multiple directions)}}|\,du\,dv$

How many coordinates do we need to specify a point on a 2D surface?

- 2 coordinates:
  - u up and down
  - v left and right

$$\vec{r}(u,v) = \underbrace{f(u,v)}_{x}\,\hat{i} + \underbrace{g(u,v)}_{y}\,\hat{j} + \underbrace{h(u,v)}_{z}\,\hat{k}$$

$$d\sigma = \underbrace{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}_{\text{area of this parallelogram}}\,du\,dv$$



Let's tackle the first problem: How to express a surface?

Plane curves:

(1) Parametrized curves

$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j}$

$a \le t \le b$

(2) Implicit form (level curves)

$F(x,y) = c$

Ex: $F(x,y) = x^2 + y^2$

then $1 = x^2 + y^2$ ← circle is level curve

(3) Explicit form

$y = f(x)$

used arc length differential

Surfaces:

(1) Parametrized surface

$\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$

$(u,v) \in R$

R in uv-plane

often: R will be box

$a \le u \le b$

$c \le v \le d$

(2) Implicit form (level surfaces)

$F(x,y,z) = c$

Ex: $F(x,y,z) = x^2 + y^2 + z^2$

then $4 = x^2 + y^2 + z^2$ ← sphere is level surface

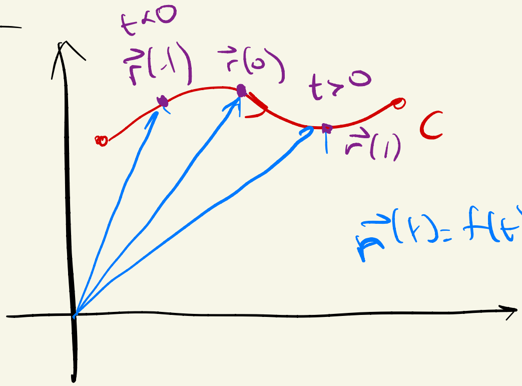
(3) Explicit form:

$z = f(x,y)$

something like arc length differential

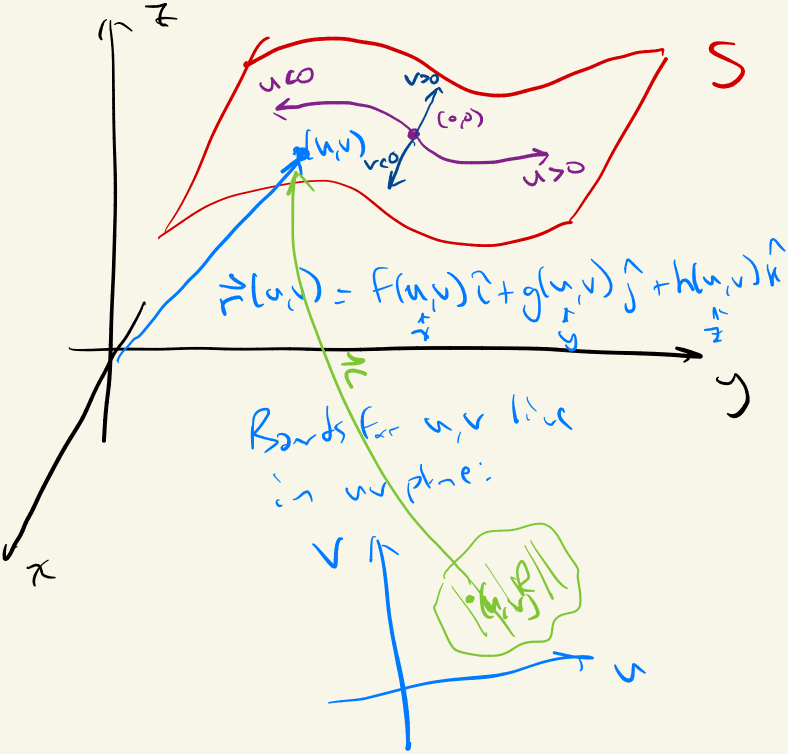
① Parameterized Surfaces: An art, not a science

1D: Given a parametrization $\vec{r}(t)$, can move up and down on a curve by just varying t:



$$\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$$

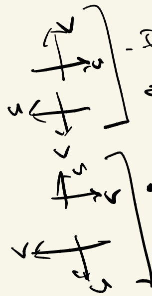
Now, 2 coordinates: forwards + backwards, left & right:



$$\vec{r}(u,v) = \underset{x}{f(u,v)}\,\hat{i} + \underset{y}{g(u,v)}\,\hat{j} + \underset{z}{h(u,v)}\,\hat{k}$$

Bounds for u, v live in uv plane:

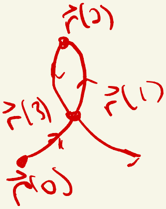
orientation = choosing which increasing directions for both u and v.



- If right-handed, orientation +
- If left-handed, orientation −

Notice from 1D:

For $\vec{r}(t)$ to be simple...



Don't want it to cross itself in this class.
Need: $\vec{r}(t)$ to be 1-to-1

If we don't want surface touching itself – require
$\vec{r}(u,v)$ is 1-to-1

Ex:

Find a parametrization of the cone

$$z=\sqrt{x^2+y^2} \qquad 0\le z\le 1$$

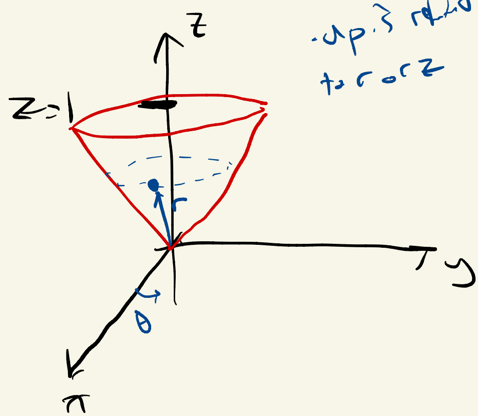
Soln:

Here, cylindrical coordinates will work

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \sqrt{x^2+y^2}$$

eqn of cone

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= r$$



up is related to r or z

only need r,θ to specify location on cone!

$$\Rightarrow \vec{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + r\,\hat{k}$$

$$\left.\begin{array}{l} 0\le r\le 1 \\ 0\le\theta\le 2\pi \end{array}\right] R$$

only one point at boundary for
probably not big deal
(inside R would be a problem)

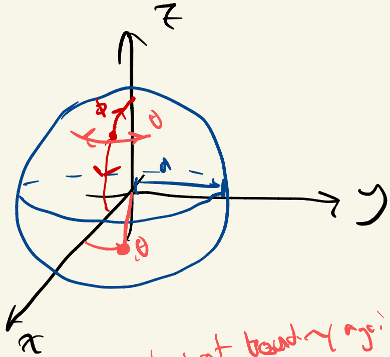
Tiny note: at $r=0$, this parametrization is not 1-to-1. All $\vec{r}(0,\theta)=(0,0,0)$
(rotation doesn't matter there).

Ex: Parametrize the sphere

$$x^2+y^2+z^2=a^2$$

Soln:

Only need to specify ϕ, θ since radius is fixed:



$$x = a\sin\phi\cos\theta$$

$$y = a\sin\phi\sin\theta$$

$$z = a\cos\phi$$

$\Rightarrow$

$$\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\hat{i} + a\sin\phi\sin\theta\,\hat{j} + a\cos\phi\,\hat{k}$$

$$\left.\begin{array}{l} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}\right\} R \text{ box}$$

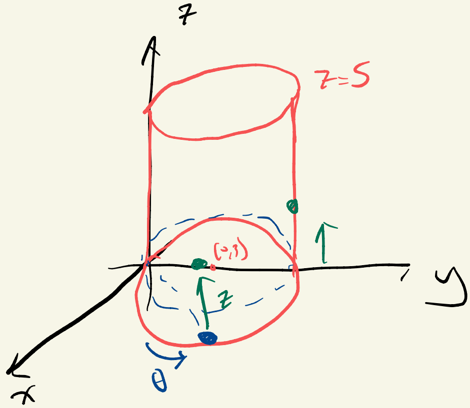
Tiny note: at boundary again, $\phi = 0$ and $\phi = \pi$, $\vec{r}$ is not one-to-one (this is ok because it's at the boundary of R)

Ex: Cylinder

$$x^2+(y-3)^2=9 \qquad 0\le z\le 5$$

Soln:

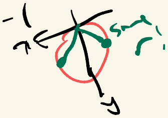
Only need rotation θ and height z to find a point. How to make this work?



$$x=r\cos\theta$$
$$y=r\sin\theta$$
$$z=z$$

Massage until x, y only depend upon θ.

(notice - if we only depended upon r, wouldn't be 1-to-1



$$x^2+(y^2-6y+9)=9$$

$$\Rightarrow \quad r^2-6r\sin\theta=0 \qquad r(r-6\sin\theta)=0$$

$$\Rightarrow \quad r=6\sin\theta \quad \text{when } r\ne 0$$

$$0\le\theta\le\pi$$

$$\Rightarrow \quad x=r\cos\theta$$

on cylinder $\rightarrow$ $\quad =6\sin\theta\cos\theta$

$$y=r\sin\theta$$
$$=6\sin\theta\sin\theta$$
$$z=z$$

all depend upon θ, z (two variables)

$$\Rightarrow \vec{r}(\theta,z)=6\sin\theta\cos\theta\,\hat{i}+6\sin^2\theta\,\hat{j}+z\hat{k}$$

$$0\le\theta\le\pi$$
$$0\le z\le 5$$

What about surface area element?

Option 1: Jacobian given

$$d\sigma = \det(\text{Jac})\, du\, dv$$

Option 2: (what we'll do in this class)

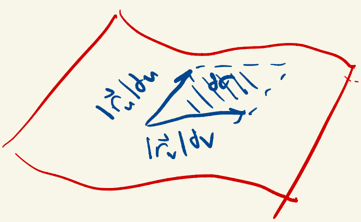
1D: $ds = |\vec{r}\,'(t)|\, dt$

↑ change in length size

Notation: $\vec{r}_u = \dfrac{\partial \vec{r}}{\partial u}$ (also works for $f(u,v)$, $f_u = \dfrac{\partial f}{\partial u}$)

$$ds = |\vec{r}_t|\, dt$$

2D:



$$\begin{aligned} d\sigma &= \text{Area parallelogram} \\ &= |\vec{r}_u\, du| \cdot |\vec{r}_v\, dv| \sin\theta \\ &= |\vec{r}_u| \cdot |\vec{r}_v| \sin\theta \; du\, dv \\ &= |\vec{r}_u \times \vec{r}_v|\, du\, dv \end{aligned}$$

* For this to work, need:

- $\vec{r}_u, \vec{r}_v$ continuous ($\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ of $\vec{r}(u,v)$ are continuous)
- $\vec{r}_u \times \vec{r}_v \neq 0$ ← if they were 0, then parallelogram

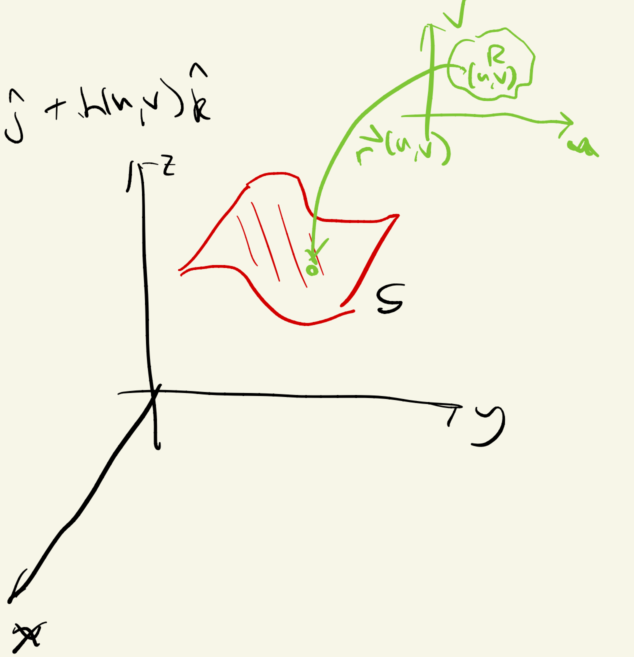
looks like no area

→ call $\vec{r}$ a smooth parametrization of our surface.

Area of a Surface:

Let $\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$

with $(u,v) \in R$. Then



$$\text{Area} = \iint_S d\sigma$$

$$= \iint_R |\vec{r}_u \times \vec{r}_v| \, dA$$

So area differential:

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

will work just like line integrals:

once we got

$$ds = |\vec{r}\,'(t)|\, dt$$

we could do line integrals!

$$\int_A^B f(x,y,z)\, ds = \int_{t=a}^{t=b} f(\vec{r}(t)) \underbrace{|\vec{r}\,'(t)|\, dt}_{ds}$$

So now: surface integrals

$$\iint_S f(x,y,z)\, d\sigma = \iint_R f(\vec{r}(u,v)) \underbrace{|\vec{r}_u \times \vec{r}_v|\, dA}_{d\sigma}$$

↑ input-plane

Calculations for surface area on extra credit tuesday!

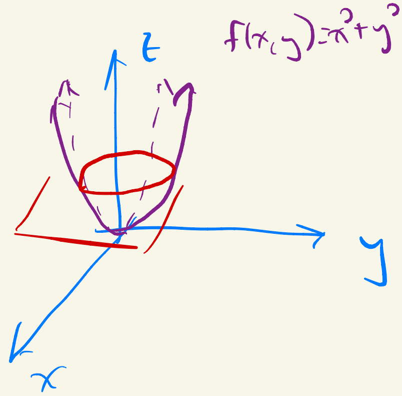
② Implicit surfaces

Recall: circles as level sets

$z=1$

$\Rightarrow$ level set $z=1$

$1 = x^2+y^2$



Likewise, we get level sets (surfaces!) for $F(x,y,z)=C$

<u>Ex:</u> $F(x,y,z)= x^2+y^2+z^2$

level surface for $1 \Rightarrow$ $1=x^2+y^2+z^2$

sphere radius 1

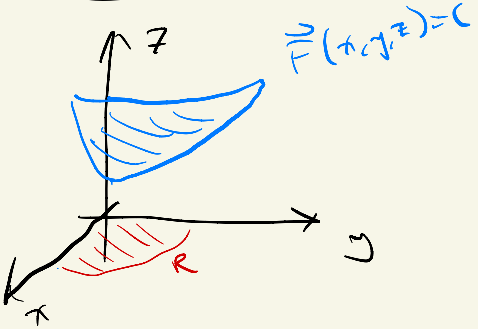
<u>Idea:</u> How to calculate $d\sigma$?

1D line integrals:

$x=t$

$f(x)=f(t)$

<u>Trick:</u> use "graph trick" to parametrize.



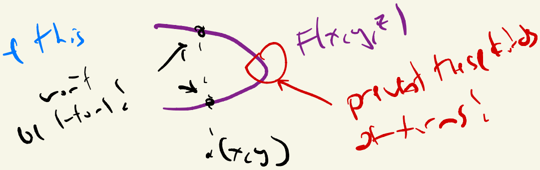
Define:

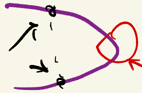
$u:=x$

$v:=y$

(*) If the surface is 1-to-1 above (x,y), Implicit function theorem guarantees there is a function $h(u,v)=z$

(*) Can't have this



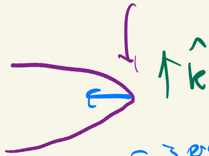


(*) to ensure this, require:

- F has continuous partials
- $\nabla F \cdot \hat{k} \neq 0$

$\hat{k}$ since above (x,y). For other e.g. (y,z), choose $\hat{i}$

Here, $\nabla F \cdot \hat{k} = 0$



∇F is perpendicular to level set $\{F(x,y,z)=c\}$

If these conditions hold,

$x = u$

$y = v$

$z = h(u,v)$

$$\vec{r}_u = \hat{i} + 0\hat{j} + \frac{\partial h}{\partial u}\hat{k}$$

$$\vec{r}_v = 0\hat{i} + \hat{j} + \frac{\partial h}{\partial v}\hat{k}$$

Implicit function thm tells us since $\nabla F \cdot \hat{k} \neq 0$

$\Leftrightarrow F_z \neq 0$

then $\dfrac{\partial h}{\partial u} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial h}{\partial v} = \dfrac{-F_y}{F_z}$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \left(\hat{i} - \frac{F_x}{F_z}\hat{k}\right) \times \left(\hat{j} - \frac{F_y}{F_z}\hat{k}\right)$$

$$= \frac{\nabla F}{F_z}$$

$$= \frac{\nabla F}{\nabla F \cdot \hat{k}}$$

works for

$dx\,dy$ xy plane $\rightarrow \hat{k} = \hat{p}$

$dy\,dz$ yz plane $\rightarrow \hat{i} = \hat{p}$

$dx\,dz$ xz plane $\rightarrow \hat{j} = \hat{p}$

$$\Rightarrow \quad d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \frac{|\nabla F|}{|\nabla F \cdot \hat{p}|}\, dA$$

works for
dxdy xy plane $\to \hat{k} = \hat{p}$
dydz yz plane $\to \hat{i} = \hat{p}$
dxdz xz plane $\to \hat{j} = \hat{p}$

Surface Area Formula for Implicit Surfaces

Area of surface $F(x,y,z) = c$ where F is smooth and R closed and bounded region R is

$$\text{Area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{p}|}\, dA$$

where $\hat{p}$ is perpendicular to area element dA and $\vec{\nabla F} \cdot \hat{p} \neq 0$ on R

Ex: EC. Tuesday

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③ Explicit form

If $z = f(x,y)$, can use parametric form of graph

let $\quad x = u$
$\quad\quad y = v$

$$\Rightarrow \quad \vec{r}(u,v) = u\hat{i} + v\hat{j} + f(u,v)\hat{k}$$

$$\Rightarrow \quad \vec{r}_u = \hat{i} + 0\hat{j} + f_u\hat{k}$$

$$\Rightarrow \quad \vec{r}_v = 0\hat{i} + \hat{j} + f_v\hat{k}$$

$$\Rightarrow \quad \vec{r}_u \times \vec{r}_v = -f_u\hat{i} - f_v\hat{j} + 1\hat{k}$$

$$\Rightarrow \quad d\sigma = \sqrt{f_u^2 + f_v^2 + 1}\; du\, dv$$

$$= \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy$$

Surface area of a graph $z = f(x,y)$

If $z = f(x,y)$ over a region R with f having continuous partials

$$\text{Surface Area} = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy$$

Ex on EC Tuesday